Parametric resonances in synchrotrons with two rf systems

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When a double rf system is subjected to sinusoidal phase modulation, the Poincaré surfaces of the section display a rich spectrum of resonance islands. Stable and unstable fixed points of these resonance islands form a tree of bifurcation branches which can be explained as parametric resonances generated by external phase modulation. A semianalytic determination of the condition for the bifurcation of fixed points is presented for an autonomous Hamiltonian of one degree of freedom with sinusoidal time dependent perturbation.

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For low-energy synchrotrons, a charged particle in a bunched beam may encounter enormous electromagnetic forces. The effects of the space charge force are manifested as transverse incoherent and coherent tune shifts, longitudinal impedance, and potential well distortions, etc. The beam intensities in booster synchrotrons at Fermilab, Brookhaven National Laboratory, and Conseil Europeén pour la Recherche Nucléaire (CERN) are known to be limited by the incoherent space charge tune shift [1]. To improve beam intensity for the CERN low-energy boosters, two rf cavities operating at harmonics 5 and 10 have been used to flatten out the longitudinal bunch shape. This improves the beam intensity by about 25–30% [2]. However, the system exhibited coherent sextupole and decapole synchrotron mode instabilities [3]. Microwave instabilities can result from insufficient Landau damping due to a small local synchrotron tune spread [4]. But, since the observed instabilities were found to be independent of the beam intensity, they might arise from single particle dynamics associated with intrinsic time dependent noise in the accelerator.

Recently a class of low-energy synchrotrons with electron cooling and/or stochastic cooling have been constructed for research in nuclear and atomic physics [5]. These cooler rings also encounter an insurmountable space charge problem related to the high charge density attained by electron cooling. The beam intensity of these cooler rings has been found to be operating at the boundary of longitudinal and transverse stabilities [6]. To achieve the high intensity needed for nuclear and atomic physics experiments, it is a logical step to employ a double rf system in order to stretch out the longitudinal profile. In our first experiment with a double rf system at the Indiana University Cyclotron Facility cooler ring, the bunched beam intensity was found to increase by about a factor of 4 in comparison with that achieved in operating only the main rf cavity at an identical rf voltage.

The double rf system is generally important for many synchrotrons which require bunched beam manipulation. Furthermore, there have been recent studies on stochastic cooling with two rf systems [7]. These studies were based on a first order perturbation expansion of the double rf potential, where an analytic solution is available. However, the synchrotron tune obtained from the first order perturbation theory is not reliable for phase amplitudes beyond about 1 rad. Thus, the response of the system to an external time dependent perturbation is considerably different than that of the first order perturbation theory. The responses to these perturbations are characterized by their resonance behavior and are called parametric resonances. This paper is organized as follows. We will formulate the unperturbed Hamiltonian for the double rf system, define the action-angle variables, and evaluate the synchrotron tune as a function of action (or energy). The time dependent perturbation such as the rf phase modulation is then introduced and expanded in terms of the action-angle variables. The parametric resonances arising from the time dependent perturbation are then identified. We then compare our theory with experimental data for a single rf system and results from numerical simulations for the double rf system.

For an ideal synchronous particle orbiting in a circular accelerator at the angular revolution frequency ω_0, the rf accelerating field is operating at a harmonic of the revolution frequency. The ratio of the rf frequency to the revolution frequency is called the *harmonic number* h. For a nonsynchronous particle with small momentum deviation, the rf sinusoidal field also provides a focusing force. Thus, nonsynchronous particles are executing synchrotron oscillations about the synchronous particle at a frequency called the *synchrotron frequency.* The number of synchrotron oscillations in one orbital revolution is called the *synchrotron tune* Q_s.

For a double rf system, let h_1, h_2 be harmonic numbers, V_1, V_2 be voltages of the primary and the secondary rf cavities, respectively. We consider the stationary state case so that the synchronous particle does not gain or lose energy in either cavity. Let ν_s be the small amplitude synchrotron tune with the primary rf system alone, i.e., $\nu_s = \left(\frac{h_1 e V_1 |\eta|}{2\pi\beta^2 E}\right)^{1/2}$, where βc and E are, respectively, the speed and the total energy of the particle, and η is the phase slip factor. Hamilton's equations for single particle synchrotron motion are

$$\dot{\phi} = \nu_s \delta, \quad \dot{\delta} = -\nu_s(\sin\phi - r\sin h\phi).$$

Here the dots are derivatives with respect to the orbiting angle θ, which serves as the time coordinate, (ϕ, δ) are the normalized conjugate phase space variables referenced to the primary rf system with $\delta = \frac{h_1\eta}{\nu_s}\frac{\Delta p}{p}$ as the normalized off momentum variable, $h = \frac{h_2}{h_1}$ is the ratio of the harmonic numbers, and $r = \frac{V_2}{V_1}$ is the ratio of rf voltages. The corresponding Hamiltonian is $H_0 = \frac{1}{2}\nu_s\delta^2 + V(\phi)$ with $V(\phi) = \nu_s\left[(1-\cos\phi) - \frac{r}{h}(1-\cos h\phi)\right]$. Although the method we use can be applied for arbitrary r and h, we consider only the case $r = \frac{1}{h}$. For the purpose of illustration, numerical evaluation will be done for $h = 3$.

The action is given by

$$J = \frac{1}{2\pi}\oint \delta d\phi, \tag{1}$$

and the phase space area of a given torus is $2\pi J$. Since the Hamiltonian is autonomous, a torus corresponds to a constant "energy", i.e., $H_0 = E$, which, for a stable orbit ($0 \le \frac{E}{\nu_s} \le \frac{16}{9}$), can be fitted numerically with the following expression, $\frac{E}{\nu_s} = AJ^{4/3}(1 - a_1J^{2/3} + a_2J^{4/3} - a_3J^2)$ ($J \le 2.280$), where the parameter $A = \left[\frac{3^{3/4}\pi}{4K}\right]^{4/3}$ is obtained from the first order perturbation expansion in the potential, i.e., $V \approx \frac{1}{3}\phi^4$, with the complete elliptical function of first kind $K = K(\frac{1}{2}) = 1.854\,07$. Applying the Bogoliubov averaging method [8] to the double rf potential, $V(\phi)$, one obtains $a_1 = 0.1762$ and $a_2 = 0.0424$. Since the rate of convergence in the E vs J expansion is very slow for a large J, the truncated a_3 term is fitted to the numerical solution of Eq. (1) in order to duplicate the characteristic behavior of the synchrotron tune $Q_s = \frac{dE}{dJ}$. This results in $a_3 \approx 0.039$. Figure 1 shows Q_s/ν_s and its rational multiples as a function of the energy. The derivative of the synchrotron tune with respect to the action becomes zero at $J \approx 1$ rad^2, where Landau damping, an essential mechanism for beam stability, also disappears.

To study the particle beam stability, we apply a small perturbation to the system and measure the response of the particle motion. We consider here a small perturbation produced by external phase modulation, where the equation for the phase ϕ is replaced by, $\dot{\phi} = \nu_s\delta + \nu_m a\cos\nu_m\theta$. Here ν_m and a are, respectively, the modulation tune and the modulation amplitude. Such an external modulation may arise from synchrobetatron coupling, rf noise, or a wake field resulting from longitudinal impedances, etc. [9]. The corresponding Hamiltonian becomes

$$H = H_0 + \nu_m a\delta\cos\nu_m\theta. \tag{2}$$

In the limit of small perturbation, i.e., $a \ll 1$, the solution can be expanded in terms of the action angle of the unperturbed Hamiltonian H_0. Using the generating function $F_2(\phi, J) = \int_{\hat{\phi}}^{\phi}\delta(\phi')d\phi'$, where $\hat{\phi}$ is an extremum

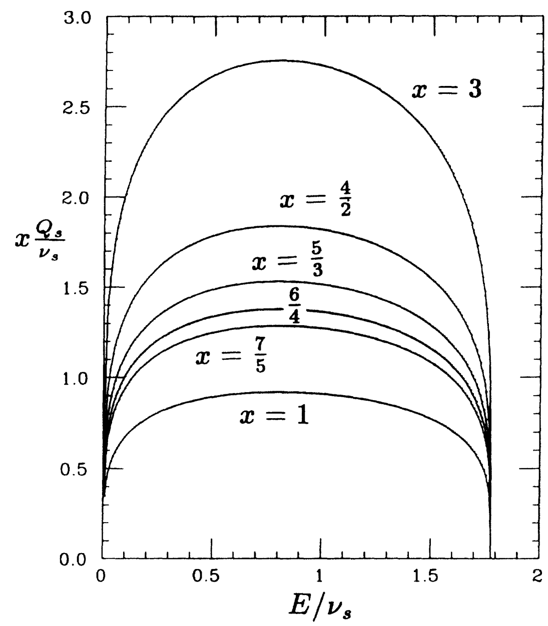

FIG. 1. The synchrotron tune $\frac{Q_s(J)}{\nu_s}$ and some of its rational harmonics for $r = \frac{1}{h}$, $h = 3$ are shown as a function of "energy."

of the phase angle for a given torus, the angle variable is then given by $\psi = \frac{\partial F_2}{\partial J} = \frac{\partial E}{\partial J}\int_{\hat{\phi}}^{\phi}\frac{\partial\delta}{\partial E}d\phi'$.

Now the task is to express the perturbation in terms of the action angle of the unperturbed Hamiltonian, i.e. $\delta = \sum_n g_n(J)e^{in\psi}$ [9]. Here the expansion amplitude $g_n(J)$ can be obtained from the inverse Fourier transform as

$$g_n(J) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta e^{-in\psi}d\psi, \tag{3}$$

which can be evaluated and parametrized in terms of J. Since $V(\phi)$ is an even function, ψ varies from $-\pi$ to π for a given torus, and δ is an odd function of ψ with reflective symmetry about the δ axis in the (ϕ, δ) phase space, the integral of Eq. (3) is zero except for odd integral n. Thus, *phase modulation of the double rf system gives rise to only odd order excitations*, similar to that of the single rf system. For small amplitude oscillations in the single rf system, the dominant resonance driving term is the first harmonic [9]. On the other hand, the resonance driving strengths for the double rf system are given by $|g_{2n+1}| \approx 0.8(2n+1)e^{-n\pi}J^{2/3}$ $(n = 1, 3, 5, \ldots)$, so the resonance strength is distributed over many harmonics.

Once the g_n coefficients are obtained, the Hamiltonian of Eq. (2) becomes

$$H = E(J) + \nu_m a\sum_{n=odd}|g_n(J)|[\cos(n\psi - \nu_m\theta + \gamma_n) + \cos(n\psi + \nu_m\theta + \gamma_n)], \tag{4}$$

where γ_n is the phase of the Fourier amplitude g_n. For $a \ll 1$, we have $\dot{\psi} \approx Q_s$ and the resonance (stationary phase) condition occurs when the modulation tune equals an odd integral multiple of the synchrotron tune. In a single resonance dominated regime, we transform the coordinates into the resonance rotating frame by using the generating function $F_2(\psi, I) = (\psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n})I$. The new conjugate action-angle variables (I, χ) are given by $I = J$, $\chi = \psi - \frac{\nu_m}{n}\theta + \frac{\gamma_n}{n}$, and the Hamiltonian becomes

$$H = E(I) - \frac{\nu_m}{n} I + \nu_m a |g_n(I)| \cos(n\chi) + \Delta H(I, \chi, \theta). \tag{5}$$

Neglecting the time dependent perturbation with $\langle \Delta H(I, \chi, \theta) \rangle = 0$, the stable and unstable fixed points (SFP's and UFP's) are given by

$$\sin n\chi_{\rm fp} = 0, \quad nQ_s(I_{\rm fp}) - \nu_m \pm n\nu_m a |g_n'(I_{\rm fp})| = 0, \tag{6}$$

and the Poincaré surfaces of section around SFP's are composed of n islands. The width of a resonance island is approximately given by $\Delta I \approx 4 \left[\frac{\nu_m a |g_n|}{|\frac{\partial Q_s}{\partial I}|}\right]^{1/2}_{I=I_{\rm sfp}}$. When the resonance action is near $I_{\rm sfp} \approx 1$ rad^2, where the detuning parameter $|\frac{\partial Q_s}{\partial I}|$ is small, the island width becomes large.

We will now discuss the bifurcation of fixed points for the Hamiltonian of Eq. (2). The invariant tori for the Hamiltonian can be obtained by numerically integrating Hamilton's equations of motion, and by taking the Poincaré surfaces of section. Figure 2 shows examples of these Poincaré surfaces of section with $\nu_m = 0.5\nu_s$, $a = 2.5°$, and $\nu_s = 0.0008$, where the absolute value of ν_s is irrelevant to the dynamics of the system. The stochasticity at the origin and the separatrix arises from overlapping resonances, which can be understood by drawing a horizontal line at $\nu_m = 0.5\nu_s$ in Fig. 1. Resonances will occur at the energies where the line cut through resonance curves $Q_s(I), \frac{7}{5}Q_s(I), \frac{6}{4}Q_s(I), \frac{5}{3}Q_s(I), \frac{4}{2}Q_s(I), 3Q_s(I)$, etc. The chaotic region near the origin arises from overlapping higher order resonance islands occurring in the order of $\frac{4}{2}, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}$. Here, the $\frac{4}{2}$ (or 4:2) resonance arises from second order perturbation by combining the $n = 1$ (1:1) and $n = 3$ (3:1) harmonics; the $\frac{5}{3}$ resonance arises from third order perturbation by combining 1:1 and 4:2 resonances, etc. The sea of stochasticity arises from overlapping separatrices of these high order resonances. At a modulation amplitude $a \leq 0.5°$, higher order resonances become invisible, while the $n = 1, 3, 5, \ldots$, resonances remain important.

Since the synchrotron tune peaks at $\hat{Q}_s \approx 0.9177\nu_s$ and vanishes at both large and small actions or energies, the resonance condition of Eq. (6) for a given order n will occur at two different actions until the modulation tune reaches the peak synchrotron tune $\hat{Q}_s$. Therefore, when ν_m is increased toward $\hat{Q}_s$, the SFP and the UFP associated with the outer amplitude and the SFP and the UFP associated with the inner amplitude approach each other. The inner SFP and the outer UFP form a bifurcation branch around $\nu_m = \hat{Q}_s$. Similarly, the inner UFP and the outer SFP form another bifurcation branch. When the modulation tune is increased beyond the first order resonance, higher order resonances bifurcate in a similar fashion. This tree of bifurcation continues until the driving amplitude g_n becomes too small to be detected.

Figure 3 shows a compilation of measured SFP's for the single rf system subject to phase modulation [9], where the phase modulation amplitudes were $a = 0.57°, 1.14°, 2.29°$, and $3.36°$, respectively, for the first harmonic and $a = 6°$ for the third harmonic. There was no visible second harmonic excitation due to phase modulation. The experimental data agree well with the prediction of Eq. (6). Note here that the UFP bifurcates with the inner SFP in this single rf system.

For the double rf system, the SFP's of resonance islands can be obtained from numerical tracking simulations as a function of the modulation tune (see Fig. 2). The square symbols of Fig. 4 shows the modulation tune vs the energy for SFP's obtained numerically with $a = 1°$. The harmonics of the synchrotron tune, $Q_s(E)$ and $3Q_s(E)$ as determined from Eq. (6) are shown as solid lines. We thus observe that *the bifurcation point occurs when the modulation frequency reaches the flat top of the synchrotron frequency*. When the modulation ampli-

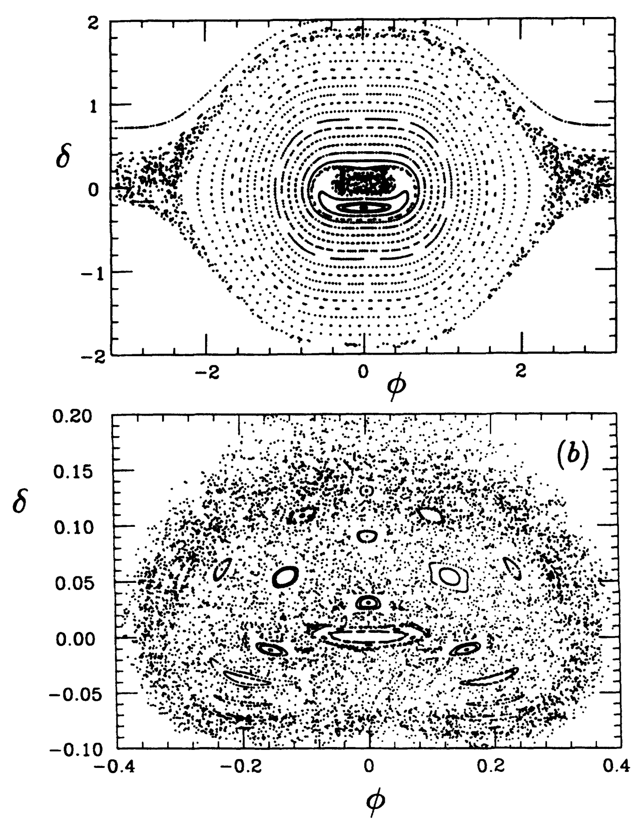

FIG. 2. The upper part of the figure shows the Poincaré surfaces of section obtained numerically with following parameters $\nu_s = 0.0008$, $\nu_m = 0.5\nu_s$, and $a = 2.5°$, where the absolute value of ν_s plays no important dynamical role. The lower part of the figure shows the close up look of the phase space map near the origin. The stochasticity near the origin and the separatrix arises from overlapping resonances. These resonances can be visualized by drawing a horizontal line at $\nu_m = 0.5\nu_s$ on Fig. 1, which cut through many resonances.

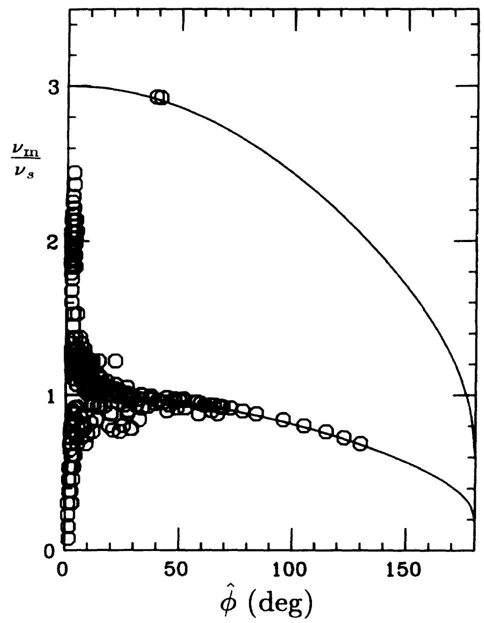

FIG. 3. The SFP's observed in a single rf system with rf phase modulation (see Fig. 3 of the first reference in Ref. [9]) as a function of the maximum phase amplitude $\hat{\phi}$ are compared with the synchrotron tune of the single rf system. The sidebands around $\nu_m = \nu_s$ arose from the 60 Hz power supply ripple.

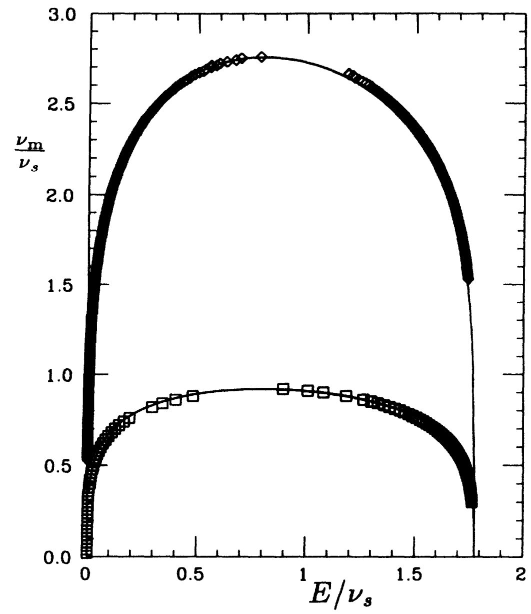

FIG. 4. The "energies" of the stable fixed points (square symbols) obtained from Poincaré surfaces of section for various modulation tunes ν_m at the modulation amplitude of $a = 1^\circ$, are compared with odd harmonics of the synchrotron tune. Note here that the bifurcation of resonance fixed points occurs at the maximum value of the synchrotron tune. The nature of the response differs greatly from that of the single rf system.

tude is increased, the branches of SFP's associated with the inner and outer amplitudes will deviate further apart from each other. The bifurcation occurs when the inner SFP's coincide with the outer UFP's and vice versa.

In conclusion, we have developed a semianalytic method for analyzing the parametric resonance of an autonomous nonlinear oscillator perturbed by a time dependent phase modulation. We found that odd order synchrotron modes are important to the double rf system subjected to sinusoidal phase modulation. Although not discussed in this paper, when the rf voltage is subjected to external sinusoidal amplitude modulation, only even order synchrotron modes are excited. Thus, the coherent beam instability observed in CERN boosters arose mainly from a perturbation having the characteristics of rf phase modulation. This correlation indicates that the nature of the coherent instability may be intimately related to resonances in the Hamiltonian dynamics.

We have also shown that the tree of bifurcation branches for the SFP's and UFP's has the characteristic tune of the Hamiltonian system. When the phase modulation amplitude is larger than 2°, the double rf system exhibits stochasticity near the origin of the phase space at small modulation tunes (see Fig.2). This chaos arises from overlapping high order resonances, which become less important at higher modulation tunes due to a smaller resonance driving strength g_n. The chaos at large J near the rf bucket boundary also arises from overlapping high order resonances, which occur in both the single rf and the double rf systems.

We have discussed only the limited parametric space of $r = \frac{1}{h}$, which is usually chosen to be the optimal operational condition for particle beam manipulation in circular accelerators. When $r > \frac{1}{h}$, the synchrotron tune ($x = 1$ in Fig. 1) would exhibit double peak structure. The tree of bifurcation for parametric resonances can be characterized with the tune of the unperturbed Hamiltonian. Three chaotic regions could exist when the phase modulation is applied. When $r < \frac{1}{h}$, the synchrotron tune becomes nonzero at the origin, and the chaos at the origin disappears. However, the system may be more susceptible to phase modulation at $\nu_m \approx \nu_s$. When $r \ll \frac{1}{h}$, then the system behaves like the single rf system, where the bifurcation of resonance islands is shown in Fig. 3. Details of these studies will be reported shortly.

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[8] In fact, the Bogoliubov averaging of the perturbed solution gives $\langle \phi^6 \rangle = 0.274\hat{\phi}^6$ and $\langle \phi^8 \rangle = 0.237\hat{\phi}^8$ instead of $\frac{5}{16}\hat{\phi}^6$ and $\frac{35}{128}\hat{\phi}^8$, where $\frac{5}{16}$ and $\frac{35}{128}$ was obtained from $\langle \cos^6 \psi \rangle$ and $\langle \cos^8 \psi \rangle$, used to obtain a_1 and a_2 in the text. However, the quality of the fit is worsened by using the Bogoliubov averaging of the perturbed solution, therefore, the numbers a_1 and a_2 should be considered as a fit to the numerical solution of $E(J)$.

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